LECTURE 6

The CLASSICAL WHITNEY PROBLEM

The Problem

Fix $m, n \geq 1$.

Given $E \subset \mathbb{R}^n$ compact

and given $\varphi : E \to \mathbb{R}$,

$$\boxed{\text{DECIDE WHETHER THERE EXISTS } F \in C^m(\mathbb{R}^n) \text{ such that } F = \varphi \text{ on } E.}$$

If such an F exists,

then we ask:

How small can we take $\|F\|_{C^m(\mathbb{R}^n)}$?

What can we say about

$\partial^\alpha F(x_0)$ $(|\alpha| \leq m)$

for a given point $x_0 \in E$?

Can we take F to depend

LINEARLY on φ?

ANSWER: YES

REMARKS ON THE EFFECT OF INFINITE E

Let $E \subset \mathbb{R}^n$, $\varphi : E \to \mathbb{R}$, $M > 0$ be given.

$$\boxed{\text{Suppose we know that } F \in C^m(\mathbb{R}^n) \text{ with norm } O(M) \text{ and } F = \varphi \text{ on } E}$$

What can we say about the $m^{\underline{th}}$ DERIVATIVES of F at a given point of E?

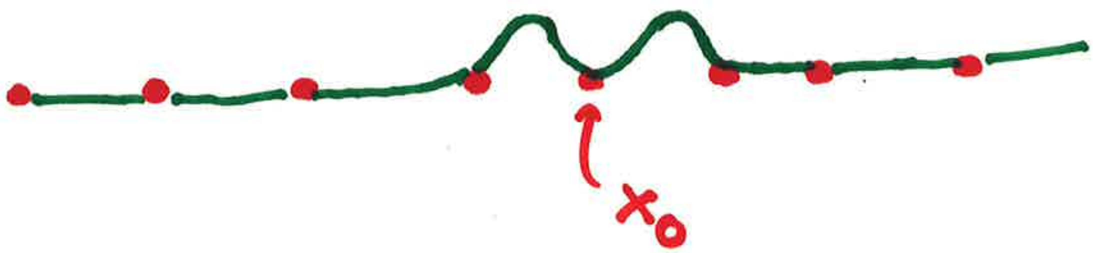
ANSWER

If E is finite, we can say NOTHING beyond the obvious fact that those derivatives are $O(M)$.

If E is INFINITE, WE MAY BE ABLE TO SAY MUCH MORE.

SIMPLE EXAMPLES for $C^2(\mathbb{R}^1)$

1



E = SET OF RED DOTS (FINITE!)

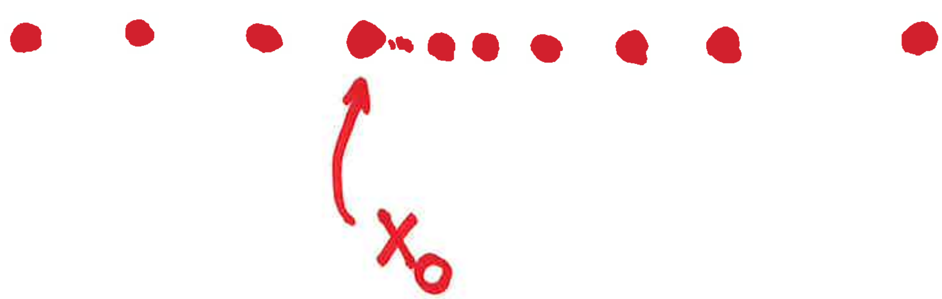
GRAPH OF F SHOWN IN GREEN

$F = 0$ on E, $\|F\|_{C^2} = O(1)$

but $F''(x_0) \sim 1$

SIMPLE EXAMPLES for $C^2(\mathbb{R}^1)$

②



E = SET OF RED DOTS (INFINITE!)

x_0 AN ACCUMULATION PT. of E

$$\boxed{\begin{gathered} F \in C^2, \quad F = 0 \text{ on } E \\ \Downarrow \\ F''(x_0) = 0. \end{gathered}}$$

To solve Whitney's Problem, we will have to take ITERATED LIMITS.

To see this, let's look at an EXAMPLE due to G. Glaeser (1958)

IN GLAESER'S EXAMPLE,

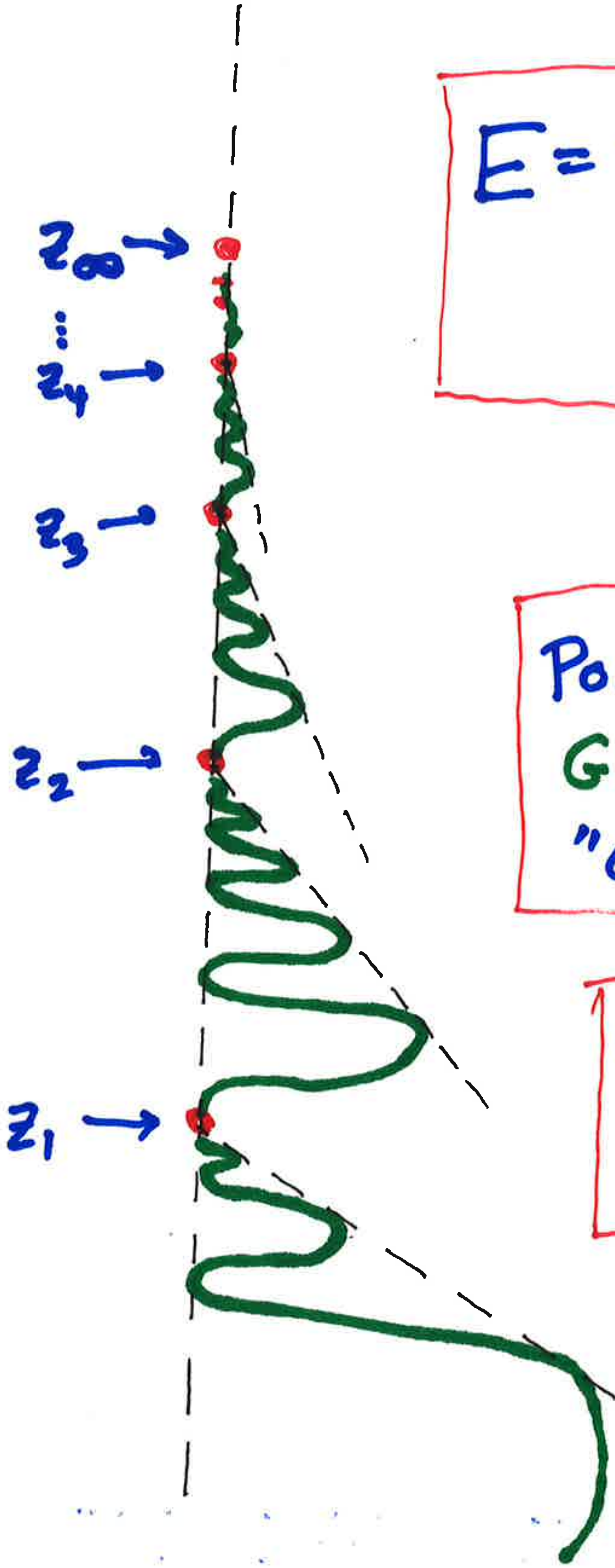
$E \subset \mathbb{R}^2$ IS A

(STRANGE-LOOKING)

ARC.

$\boxed{\text{WE ASK WHETHER A GIVEN } \varphi : E \to \mathbb{R} \text{ EXTENDS TO A FN. } F \in C^1(\mathbb{R}^2).}$

WILL NEED ITERATED LIMITS!



E = THE GREEN CURVE (INCLUDING THE RED DOTS)

POINTS SHOWN IN GREEN ARE "ORDINARY PTS"

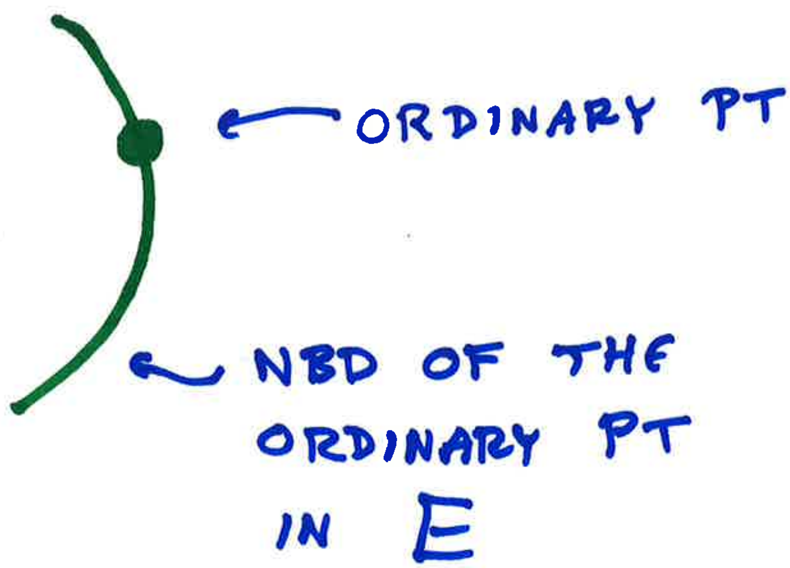
$z_1, z_2, \cdots$ ARE "SPECIAL PTS."

z_∞ IS THE "EXTRA SPECIAL PT."

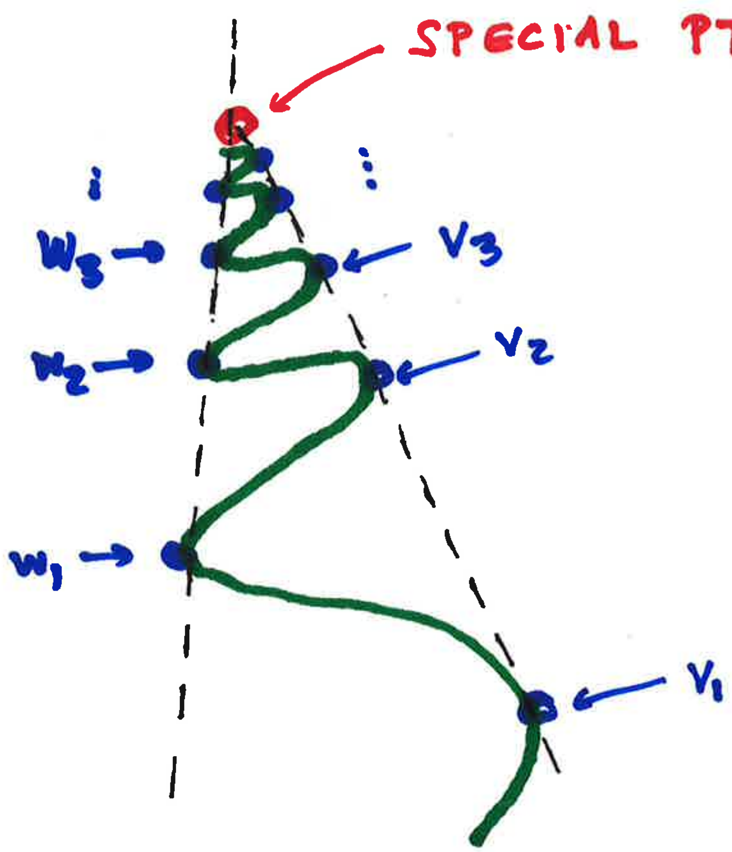
NEAR AN ORDINARY

PT OF E,

REQUIRE MERELY $\varphi \in C^1$



AT A SPECIAL PT., LIMITS MUST EXIST



DIFFERENCE QUOTIENTS INVOLVING V_i'S MUST TEND TO A LIMIT AS $i \to \infty$.

SIMILARLY FOR W_i'S.

WE RECOVER $\nabla F(z_k)$

AT THE EXTRA SPECIAL PT,

AN ITERATED LIMIT

MUST EXIST,

NAMELY

$$\lim_{K\to\infty} \nabla F(z_k).$$

RECOVER $\nabla F(z_\infty)$ as

an <u>iterated limit</u>,

but NOT as a (SINGLE) limit

of difference quotients.

That's Glaeser's Example.

Let's RETURN to the GENERAL CASE.

BUILDING ON THE WORK OF

G. GLAESER

and

E. BIERSTONE,

P. MILMAN &

W. PAWŁUCKI,

WE SOLVE WHITNEY'S PROBLEM.

To solve

WHITNEY'S PROBLEM,

WE

GENERALIZE IT.

NOTATION

Fix $m, n \geq 1$.

We work in $C^m(\mathbb{R}^n)$, the space of all $F : \mathbb{R}^n \to \mathbb{R}$ with continuous & bounded derivatives up to order m.

$$\|F\|_{C^m(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \leq m} |\partial^\alpha F(x)|.$$

For $F \in C^m(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$J_x(F)$ DENOTES THE

m th DEGREE TAYLOR POLY.

OF F AT x.

NOTE: m^{th} DEGREE, NOT $(m-1)^{\text{rst}}$.

BECAUSE E IS INFINITE, THE HIGHEST ORDER DERIV'S WILL PLAY A RÔLE

WE WRITE $\mathcal{P}$ TO DENOTE THE VECTOR SPACE OF ALL (REAL) $m^{\underline{th}}$ DEGREE POLYS ON $\mathbb{R}^n$.

Thus, $J_x(F) \in \mathcal{P}$.

For $x \in \mathbb{R}^n$ there's a natural multiplication on $\mathcal{P}$ ("JET MULTIPLICATION"), DENOTED $\odot_x$, AND DEFINED BY

$$P \odot_x Q = J_x(PQ).$$

The point is that

$$J_x(FG) = J_x(F) \odot_x J_x(G)$$

for $F, G \in C^m(\mathbb{R}^n)$.

$\boxed{\text{The RING OF JETS AT } x, \text{ denoted } \mathcal{R}_x, \text{ is the vector space } \mathcal{P} \text{ equipped with the MULTIPLICATION } \odot_x}$

TO GENERALIZE WHITNEY'S PROBLEM, WE INTRODUCE THE NOTION OF A BUNDLE.

A **BUNDLE** is a collection

$\mathcal{H} = (H(x))_{x \in E}$

of affine subspaces $H(x) \subset \mathcal{P}$,

indexed by the points x

of a compact subset $E \subset \mathbb{R}^n$,

such that ...

DEF. OF BUNDLES (CONTINUED)

for each $x \in E$,

EITHER

$$\boxed{\begin{array}{l} H(x) = \emptyset \\ \left\{ \text{EMPTY SET — ALLOWED AS AN AFFINE SPACE (by CONVENTION)} \right\} \end{array}}$$

OR

$$\boxed{\begin{array}{l} H(x) = f(x) + I(x) \\ \text{for some} \\ f(x) \in \mathcal{O}_x,\ I(x) \subset \mathcal{O}_x \ \underline{\underline{\text{IDEAL}}} \end{array}}$$

WE CALL E THE **BASE**,

and WE CALL $H(x)$ THE **FIBER AT x**

for the bundle $\mathcal{H} = (H(x))_{x \in E}$.

$$\boxed{\text{We make NO ASSUMPTION on the dependence of } H(x) \text{ on } x.}$$

For instance, $\dim H(x)$ needn't

be a MEASURABLE FN. of x.

If $\mathcal{H} = (H(x))_{x \in E}$

and $\mathcal{H}' = (H'(x))_{x \in E}$

are bundles over the same base E, then

$\mathcal{H}'$ is a SUBBUNDLE of $\mathcal{H}$

(WRITTEN $\mathcal{H} \supset \mathcal{H}'$)

PROVIDED

$H(x) \supset H'(x)$ for all $x \in E$.

Allowing the EMPTY SET $\emptyset$ AS A FIBER SEEMS STRANGE.

WE'LL SOON SEE WHY IT'S A GOOD IDEA.

Let $\mathcal{H} = (H(x))_{x \in E}$

be a bundle.

A <u>SECTION</u> of $\mathcal{H}$ is

a function $F \in C^m(\mathbb{R}^n)$

such that

$J_x(F) \in H(x)$ for all $x \in E$.

Obviously, that can't happen

if $\mathcal{H}$ has any empty fibers.

WE CAN NOW STATE OUR

GENERALIZED

WHITNEY

PROBLEMS.

Given a bundle $\mathcal{H}$,

DECIDE WHETHER $\mathcal{H}$

HAS A SECTION.

IF $\mathcal{H}$ has a section F,

then how small can we

take $\|F\|_{C^m(\mathbb{R}^n)}$?

Let $\mathcal{H} = (H(x))_{x \in E}$ be a bundle.

Fix $x_0 \in E$ and $P_0 \in H(x_0)$.

$$\boxed{\text{Decide whether } \mathcal{H} \text{ has a section } F \text{ such that } J_{x_0}(F) = P_0.}$$

THE CLASSICAL WHITNEY PROBLEMS ARE SPECIAL CASES OF THE ABOVE "GENERALIZED WHITNEY PROBLEMS"

LET'S SEE WHY ...

Let $E \subset \mathbb{R}^n$ be compact,

and let $\varphi : E \to \mathbb{R}$ be given.

$$\boxed{\text{We want to know whether there exists } F \in C^m(\mathbb{R}^n) \text{ such that } F = \varphi \text{ on } E.}$$

DEFINE A BUNDLE $\mathcal{H} = (H(x))_{x \in E}$ as follows:

For each $x \in E$, set

$$\boxed{f(x) = \left[\begin{array}{l}\text{The constant poly.} \\ \text{whose value everywhere} \\ \text{is } \varphi(x)\end{array}\right]}$$

$$\boxed{\begin{array}{l} I(x) = \{P \in \mathcal{P} : P(x) = 0\} \\ \text{(an IDEAL in } \mathcal{R}_x) \end{array}}$$

$$\boxed{H(x) = f(x) + I(x)}$$

So $H(x) = \{ P \in \mathcal{P} : P(x) = \varphi(x) \}$.

Then $\mathcal{H} = (H(x))_{x \in E}$

is a <u>BUNDLE</u>, and

$\boxed{\text{a SECTION OF } \mathcal{H} \text{ IS PRECISELY A FUNCTION } F \in C^m(\mathbb{R}^n) \text{ SUCH THAT } F = \varphi \text{ on } E.}$

For this bundle $\mathcal{H}$,

$$\boxed{\text{Does } \mathcal{H} \text{ have a section?}}$$

Means

$$\boxed{\begin{array}{l}\text{Is there an } F \in C^m(\mathbb{R}^n) \\ \text{such that } F = \varphi \text{ on } E\text{?}\end{array}}$$

For this bundle $\mathcal{H}$,

$$\boxed{\text{HOW SMALL CAN WE TAKE THE NORM OF A SECTION of } \mathcal{H}\text{?}}$$

MEANS

$$\boxed{\text{How small can we take the } C^m \text{ norm of a fn } F \text{ s.t. } F = \varphi \text{ on } E\text{?}}$$

For this bundle,

$$\boxed{\text{Does } \mathcal{H} \text{ have a section } F \text{ s.t. } J_{x_0}(F) = P_0 \text{ ?}}$$

MEANS

$$\boxed{\text{Does there exist } F \in C^m(\mathbb{R}^n) \text{ s.t. } F = \varphi \text{ on } E \text{ and } J_{x_0}(F) = P_0 \text{ ?}}$$

So, as promised,

our GENERALIZED WHITNEY PROBS. FOR BUNDLES

include the classical

Whitney problems

as special cases.

There's also a BUNDLE version of the classical question

"Can we take F to depend linearly on φ ?"

but we won't discuss it here, except to say that the answer is

YES!

WE PREPARE TO SOLVE THE GENERALIZED WHITNEY PROBS. by introducing the idea of

$\boxed{\text{GLAESER REFINEMENTS.}}$

Given a bundle $\mathcal{H}$,

we will define

another bundle $\tilde{\mathcal{H}}$,

called the

GLAESER REFINEMENT of $\mathcal{H}$,

in such a way that

three key properties hold:

PROPERTIES OF THE GLAESER REFINEMENT

Let $\tilde{\mathcal{H}}$ be the Glaeser refinement of $\mathcal{H}$.

Then

$\tilde{\mathcal{H}}$ IS A SUBBUNDLE of $\mathcal{H}$,

~~and~~ but

Every section of $\mathcal{H}$ is also a section of $\tilde{\mathcal{H}}$.

MOREOVER,

$\boxed{\begin{array}{l}\tilde{\mathcal{H}} \text{ CAN BE COMPUTED FROM } \mathcal{H} \\ \text{by doing LINEAR ALGEBRA} \\ \text{(in fixed dimension)} \\ \text{and taking a limit.}\end{array}}$

To motivate the definition of the Glaeser refinement, we recall a simple corollary of Taylor's thm:

Let $F \in C^m(\mathbb{R}^n)$ and let $x_0 \in \mathbb{R}^n$.

Given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x', x'' \in B(x_0, \delta)$, the jets $P' = J_{x'}(F)$ and $P'' = J_{x''}(F)$ satisfy

$$|\partial^\alpha (P' - P'')(x')| \le \varepsilon |x' - x''|^{m - |\alpha|}$$

for $|\alpha| \le m$.

Now we define the Glaeser refinement $\tilde{\mathcal{H}} = (\tilde{H}(x))_{x \in E}$ of a bundle $\mathcal{H} = (H(x))_{x \in E}$.

Fix a large enough integer constant k, DEPENDING only on m, n.

For each $x_0 \in E$, $\tilde{H}(x_0)$ is defined to consist of all $P_0 \in H(x_0)$ for which the following holds:

DEFINING CONDITION FOR $P_0 \in \tilde{H}(x_0)$:

GIVEN $\varepsilon > 0$ there exists $\delta > 0$

s.t.

for all $x_1, \dots, x_k \in E \cap B(x_0, \delta)$

there exist

$P_1 \in H(x_1), \dots, P_k \in H(x_k)$

s.t.

$$|\partial^\alpha (P_i - P_j)(x_i)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|}$$

for $|\alpha| \leq m$ and $i, j = 0, 1, \dots, k$

IN A MOMENT, WE WILL CHECK THAT THE GLAESER REFINEMENT HAS THE 3 KEY PROPERTIES ASSERTED ABOVE, but first we make a few REMARKS.

① NOTE THAT $P_0, P_1, \dots, P_k$

ALL ENTER INTO THIS DEFINITION,

BUT P_0 PLAYS A RÔLE

DIFFERENT FROM THAT OF $P_1, \dots, P_k$.

MORE PRECISELY, P_0 STAYS FIXED,

WHILE $P_1, \dots, P_k$ VARY WITH

$x_1, \dots, x_k$.

② If $x_i = x_j$ and $|\alpha| = m$,

then $|x_i - x_j|^{m-|\alpha|}$

becomes 0^0.

By convention, we take

$$|x_i - x_j|^{m-|\alpha|} \equiv 0$$

in this degenerate case.

3) The Glaeser refinement of $\mathcal{H}$ may have some EMPTY FIBERS, even if $\mathcal{H}$ has NONE.

That's why we ALLOW EMPTY FIBERS in the DEFINITION of a BUNDLE.

④ GLAESER REFINEMENTS

WILL PLAY A CRUCIAL RÔLE.

THAT'S WHY WE HAVE TO

GENERALIZE

WHITNEY'S CLASSICAL PROB'S

TO THE SETTING OF

BUNDLES.

NOW LET'S CHECK THAT GLAESER REF'MENTS HAVE THE PROMISED

$\boxed{\text{3 BASIC PROPERTIES.}}$

$\tilde{\mathcal{H}}$ IS A SUBBUNDLE of $\mathcal{H}$

Obvious!

The fiber $\tilde{H}(x_0)$ is defined to consist of all $P_0 \in H(x_0)$ such that ....

So of course

$$H(x_0) \supset \tilde{H}(x_0).$$

EVERY SECTION OF $\tilde{\mathcal{H}}$ IS ALSO A SECTION OF $\mathcal{H}$.

Let F be a section of $\mathcal{H}$,

and let $x_0 \in E$.

We must show that

$$P_0 = J_{x_0}(F)$$

belongs to $\tilde{H}(x_0)$.

That is, we must show that

$\boxed{P_0 \in H(x_0)}$ ← {YES, because F IS A SECTION OF $\mathcal{H}$}

and that

Given $\varepsilon > 0$ there exists $\delta > 0$ s.t. for all $x_1, \dots, x_k \in E \cap B(x_0, \delta)$

there exist

$P_1 \in H(x_1), \dots, P_k \in H(x_k)$ s.t.

$$\left|\partial^\alpha (P_i - P_j)(x_i)\right| \le \varepsilon \left|x_i - x_j\right|^{m-|\alpha|}$$

for $|\alpha| \le m$ and $i, j = 0, 1, \dots, k$.

(*)

Condition (*) is immediate from the Corollary of Taylor's thm mentioned a moment ago.

We just take

$$P_1 = J_{x_1}(F), \ldots, P_k = J_{x_k}(F).$$

(NOTE $P_1 \in H(x_1), \ldots, P_k \in H(x_k)$ BECAUSE F IS A SECTION OF $\mathcal{H}$.)

So, as promised,

every section of $\mathcal{H}$

is also a section of $\tilde{\mathcal{H}}$.

$\tilde{\mathcal{H}}$ CAN BE COMPUTED FROM $\mathcal{H}$

by doing LINEAR ALGEBRA

(in fixed dimension)

and taking a LIMIT.

Fix $x_0 \in E$ and $P_0 \in H(x_0)$.

WE DECIDE WHETHER $P_0 \in \tilde{H}(x_0)$

AS FOLLOWS:

For $x_1, \dots, x_k \in E$ and

$(P_1, \dots, P_k) \in H(x_1) \oplus \cdots \oplus H(x_k)$,

SET

$$\boxed{Q(P_1, \dots, P_k \mid x_1, \dots, x_k) = \sum_{i,j=0}^{k} \sum_{|\alpha| \le m} \left[\frac{\partial^\alpha (P_i - P_j)(x_i)}{|x_i - x_j|^{m-|\alpha|}} \right]^2}$$

For fixed $x_1, \dots, x_k$, the function

$(P_1, \dots, P_k) \longmapsto Q(P_1, \dots, P_k \mid x_1, \dots, x_k)$

is a quadratic function on an affine space.

Therefore,

$$Q_{min}(x_1, \dots, x_k) =$$

$$\text{MIN}\left\{ Q(P_1, \dots, P_k \mid x_1, \dots, x_k) : P_1 \in H(x_1), \dots, P_k \in H(x_k) \right\}$$

may be computed by doing LINEAR ALGEBRA IN FIXED DIMENSION.

Comparing the definitions

of Q, Q_{min}

with the definition of $\widetilde{H}(x_0)$,

WE SEE THAT

$$\boxed{\begin{array}{l} P_0 \in \widetilde{H}(x_0) \text{ if \& only if} \\ Q_{min}(x_1, \ldots, x_k) \longrightarrow 0 \\ \text{as } (x_1, \ldots, x_k) \in E \times E \times \cdots \times E \\ \text{tends to } (x_0, x_0, \ldots, x_0). \end{array}}$$

So, as promised,

we can decide whether

$\mathcal{P}_0 \in \widetilde{H}(x_0)$

by doing LINEAR ALGEBRA

in fixed dim (to compute Q_{min})

and taking a limit (of Q_{min}).

This completes the

Verification of the

3 BASIC PROPERTIES

of the

GLAESER REFINEMENT.

Using the Glaeser Refinement

We want to know whether a given bundle $\mathcal{H}_0$ has a section.

Let $\mathcal{H}_1$ be the Glaeser refinement of $\mathcal{H}_0$.

We can compute $\mathcal{H}_1$ from $\mathcal{H}_0$.

$\mathcal{H}_1$ and $\mathcal{H}_0$ have the same sections

$$\mathcal{H}_0 \supset \mathcal{H}_1$$

To pass from $\mathcal{H}_0$ to $\mathcal{H}_1$, we discard from the fibers of $\mathcal{H}_0$ certain polys that can never arise as the jet of a SECTION.

So we have MADE PROGRESS.

WE CAN KEEP GOING.

Let $\mathcal{H}_2$ be the

Glaeser refinement of $\mathcal{H}_1$,

Let $\mathcal{H}_3$ be the

Glaeser refinement of $\mathcal{H}_2$,

and so on.

WE OBTAIN BUNDLES

$\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \cdots$ SUCH THAT

$$\boxed{\mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots}$$

yet

All the $\mathcal{H}_\ell$ have the SAME SECTIONS.

Moreover, we can compute each $\mathcal{H}_\ell$ starting from $\mathcal{H}_0$.

The $\mathcal{H}_\ell$ are called the ITERATED GLAESER REFINEMENTS of $\mathcal{H}_0$.

The Process Stabilizes

Adapting a simple, ingenious Lemma of BIERSTONE-MILMAN-PAWŁUCKI,

which in turn was adapted from a simple, ingeneous Lemma of Glaeser, we obtain the following result.

GLAESER STABILIZATION LEMMA:

Let $\ell_* = 2 \dim \mathcal{P} + 1$.

Then the iterated Glaeser

refinements satisfy

$$\mathcal{H}_{\ell_*} = \mathcal{H}_{\ell_* + 1} = \cdots$$

So there's NO POINT in

iterating more than ℓ_* times.

The proof of the Glaeser Stabilization Lemma is remarkably simple and completely elementary.

We'll give it at the end of this talk, if there's enough time.

Given a bundle $\mathcal{H} = (H(x))_{x \in E}$,

we compute its ℓ^{th} iterated

Glaeser refinement

$\mathcal{H}_\ell = (H_\ell(x))_{x \in E}$

for $\ell = 1, 2, \dots, \ell_*$.

Then

$\mathcal{H}_{\ell_*}$ may be computed from $\mathcal{H}$

$\mathcal{H}_{\ell_*}$ and $\mathcal{H}$ have the same sections

$\mathcal{H}_{\ell_*}$ is its own Glaeser ref.

WE CALL $\mathcal{H}_{l_*}$

THE STABLE GLAESER REFINEMENT of $\mathcal{H}$,

and we denote it by $\mathcal{H}_*$.

RECALL THE GENERALIZED WHITNEY PROBLEMS:

Given a bundle $\mathcal{H} = (H(x))_{x \in E}$,

does $\mathcal{H}$ have a section F?

If so, how small can we take $\|F\|_{C^m}$?

What can we say about $J_{x_0}(F)$ for a given $x_0 \in E$?

Without loss of generality,

we may replace $\mathcal{H}$ by $\mathcal{H}_{\ell_*}$.

Thus, we may assume without loss of generality that $\mathcal{H}$ is its own Glaeser ref.

We say that $\mathcal{H}$ is "Glaeser stable".

MAIN THM. :

Let $\mathcal{H} = (H(x))_{x \in E}$

be a Glaeser stable bundle.

Then $\mathcal{H}$ has a section

if & only if

all its fibers are non-empty.

<u>Corollary</u>: Let $\mathcal{H} = (H(x))_{x \in E}$ be a Glaeser stable bundle whose fibers are all non-empty.

Given $x_0 \in E$ and $P_0 \in H(x_0)$, there exists a section F of $\mathcal{H}$, such that $J_{x_0}(F) = P_0$.

The next talk will sketch the proof of the MAIN THM.

LET'S DEDUCE THE COROLLARY.

(It's EASY!)

Let $\mathcal{H} = (H(x))_{x \in E}$ be

Glaeser stable, with

non-empty fibers.

Let $P_0 \in H(x_0)$.

We define a new bundle

$\hat{\mathcal{H}}$, as follows ...

$$\hat{\mathcal{H}} = \left(\hat{H}(x) \right)_{x \in E},$$

where

$$\boxed{\begin{aligned} &\hat{H}(x) = H(x) \text{ for } x \in E \setminus \{x_0\} \\ &\hat{H}(x_0) = \{P_0\}. \end{aligned}}$$

Then $\hat{\mathcal{H}}$ is a Glaeser stable bundle with non-empty fibers.

Therefore, by the MAIN THM, $\hat{\mathcal{H}}$ has a section F.

However, a section of $\hat{\mathcal{H}}$ is precisely a fn. $F \in C^m(\mathbb{R}^n)$ s.t.

$$\boxed{\begin{array}{l} J_x(F) \in H(x) \text{ for all } x \in E \\ \text{and} \\ J_{x_0}(F) = P_0 . \end{array}}$$

THAT PROVES THE COROLLARY.

We would like to know how SMALL we can take the C^m norm of a section of $\mathcal{H}$.

This info is supplied by a FINITENESS THM for GLAESER STABLE BUNDLES, which we now present.

RECALL THAT WE HAVE FIXED A LARGE ENOUGH INTEGER CONSTANT k DEPENDING ONLY ON m, n.

Let $\mathcal{H} = (H(x))_{x \in E}$ be a bundle with non-empty fibers.

WE DEFINE THE NORM OF $\mathcal{H}$,

DENOTED $\|\mathcal{H}\|$,

TO BE THE LEAST $M \geq 0$

FOR WHICH THE FOLLOWING

HOLDS :

Given $x_1, \dots, x_k \in E$

there exist

$P_1 \in H(x_1), \dots, P_k \in H(x_k)$

such that

$$\boxed{\begin{array}{l} |\partial^\alpha P_i(x_i)| \leq M \\ \quad \text{for } |\alpha| \leq m, \ i = 1, \dots, k \\ \\ \text{and} \\ |\partial^\alpha (P_i - P_j)(x_i)| \leq M |x_i - x_j|^{m - |\alpha|} \\ \quad \text{for } |\alpha| \leq m \text{ and } i, j = 1, \dots, k. \end{array}}$$

If no such M EXISTS,

WE SAY THAT

$$\|\mathcal{H}\| = \infty.$$

Lemma:

Let $\mathcal{H}$ be a Glaeser stable bundle whose fibers are all non-empty.

Then $\|\mathcal{H}\| < \infty$.

"FINITENESS OF THE NORM"

A LITTLE LATER WE'LL SKETCH THE PROOF OF THE ABOVE LEMMA,

BUT FIRST WE ANSWER THE QUESTION:

$$\boxed{\text{How small can we take the } C^m \text{ norm of a section of } \mathcal{H}\text{?}}$$

QUANTITATIVE MAIN THM:

Let $\mathcal{H}$ be a Glaeser-stable bundle, whose fibers are all non-empty. Then $\mathcal{H}$ has a section F whose C^m-norm satisfies

$$\|F\|_{C^m(\mathbb{R}^n)} \leq C \|\mathcal{H}\|;$$

here, C depends only on m, n.

The converse inequality is obvious from Taylor's thm:

Any section F of a bundle $\mathcal{H}$ satisfies

$$\|F\|_{C^m(\mathbb{R}^n)} \geq c\,\|\mathcal{H}\|,$$

where c depends only on m, n.

So our

QUANTITATIVE MAIN THM

tells us how small we can take the C^m norm of a section of a given bundle $\mathcal{H}$.

REMARKS

Obviously, the QUANTITATIVE MAIN THM implies the MAIN THM.

I don't know how to prove the Main Thm without deducing it from the QUANTITATIVE MAIN THM.

The next lecture sketches the proof of the QUANTITATIVE MAIN THM.

LET'S NOW RETURN TO

SKETCH THE PROOF THAT

$\|\mathcal{H}\| < \infty$

for a Glaeser stable bundle

with non-empty fibers.

WE ARGUE BY CONTRADICTION

Suppose $\|\mathcal{H}\| = \infty$.

Then there exists a sequence

of k-point sets

$S^\nu = \{x_1^\nu, \dots, x_k^\nu\}$ $(\nu = 1, 2, \dots)$

and a sequence of numbers

$M_\nu \to \infty$,

such that the following holds:

PROPERTY of the S^ν, M_ν

For each ν, there <u>DO NOT</u> exist

$$P_1 \in H(x_1^\nu), \ldots, P_k \in H(x_k^\nu)$$

such that

$$\boxed{\begin{array}{l} |\partial^\alpha P_i(x_i^\nu)| \le M_\nu \\ \qquad \text{for } |\alpha| \le m \text{ and } i = 1, \ldots, k \\ \\ \text{and} \\ |\partial^\alpha (P_i - P_j)(x_i^\nu)| \le M_\nu \, |x_i^\nu - x_j^\nu|^{m-|\alpha|} \\ \qquad \text{for } |\alpha| \le m \text{ and } i, j = 1, \ldots, k. \end{array}}$$

Because E is compact,

we may pass to a subsequence,

and assume that

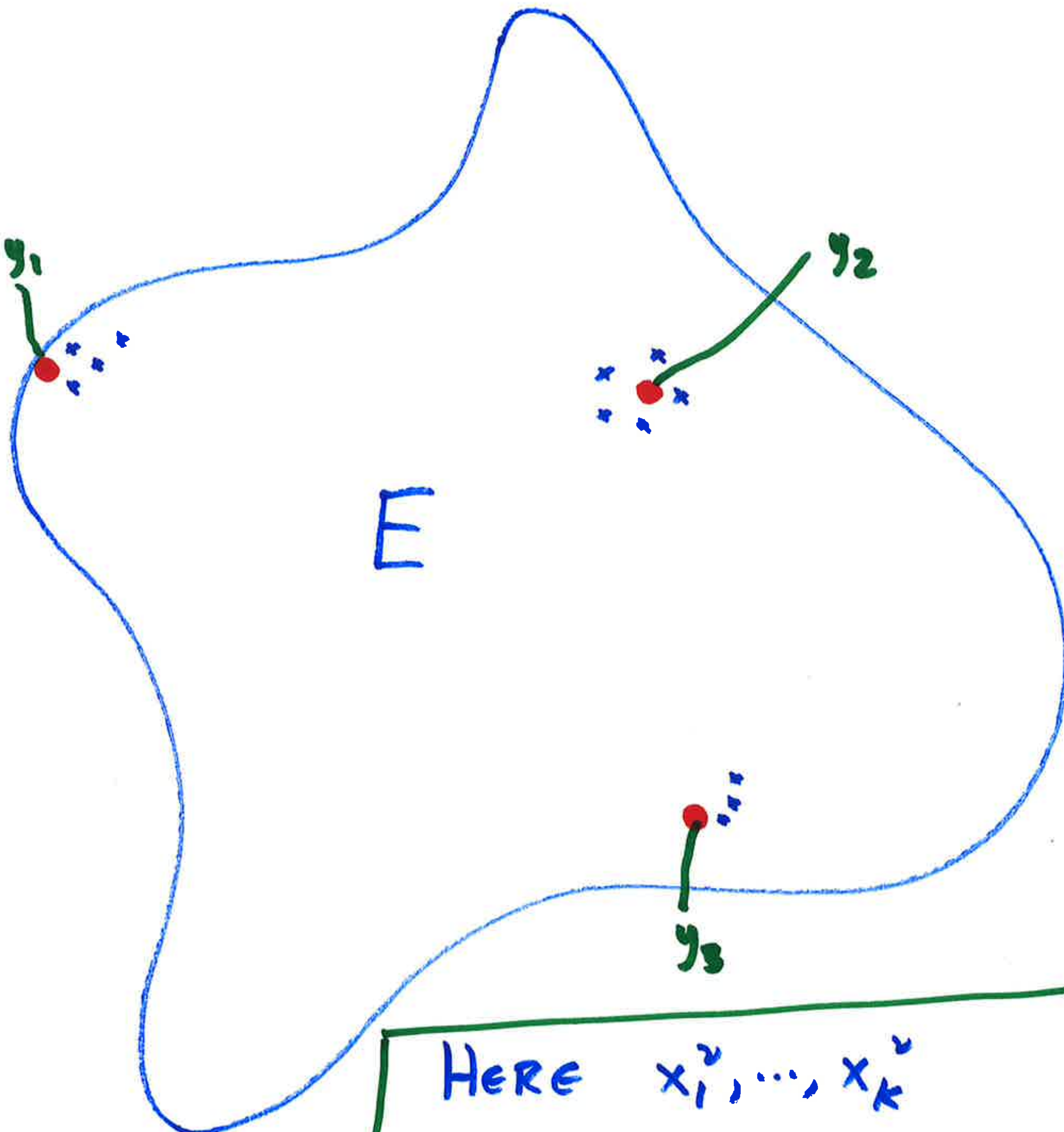
$$x_i^\nu \to x_i^\infty \text{ as } \nu \to \infty \quad (\text{each } i),$$

with each $x_i^\infty \in E$.

The x_i^∞ needn't be distinct.

Let $y_1, \dots, y_s$ be the

distinct elements of $\{x_1^\infty, \dots, x_k^\infty\}$.



Here $x_1^\nu, \dots, x_k^\nu$ are the BLUE x's and $y_1, \dots, y_s$ are the RED DOTS.

($\nu \gg 1$ FIXED)

Pick polys

$\mathring{P}_1 \in H(y_1), \ldots, \mathring{P}_S \in H(y_S)$.

(RECALL, THE FIBERS OF $\mathcal{H}$ ARE NON-EMPTY.)

BECAUSE $\mathcal{H}$ IS GLAESER-STABLE,

$$\boxed{\begin{array}{l}\mathring{P}_q \text{ belongs to the fiber at } y_q \\ \text{of the Glaeser refinement of } \mathcal{H} \\ (\text{EACH } q = 1, \dots, S)\end{array}}$$

WE APPLY THE DEF. OF THE GLAESER REFINEMENT TO $\mathring{P}_q$, TAKING (SAY) $\varepsilon = 1$.

WE LEARN THE FOLLOWING:

PROPERTY $(*)_{q}$

Given $\hat{x}_1, \dots, \hat{x}_k \in E$

sufficiently close to y_q,

there exist polys.

$\hat{P}_1 \in H(\hat{x}_1), \dots, \hat{P}_k \in H(\hat{x}_k)$

s.t.

$$\boxed{\begin{array}{l} |\partial^\alpha(\hat{P}_i - \hat{P}_j)(\hat{x}_i)| \leq |\hat{x}_i - \hat{x}_j|^{m-|\alpha|} \\ \text{for } |\alpha| \leq m, \quad i,j = 1,\dots,k\,; \\ \text{and} \\ |\partial^\alpha(\hat{P}_i - \mathring{P}_q)(y_q)| \leq |\hat{x}_i - y_q|^{m-|\alpha|} \\ \text{for } |\alpha| \leq m, \quad i = 1,\dots,k \end{array}}$$

Now let ν be large enough.

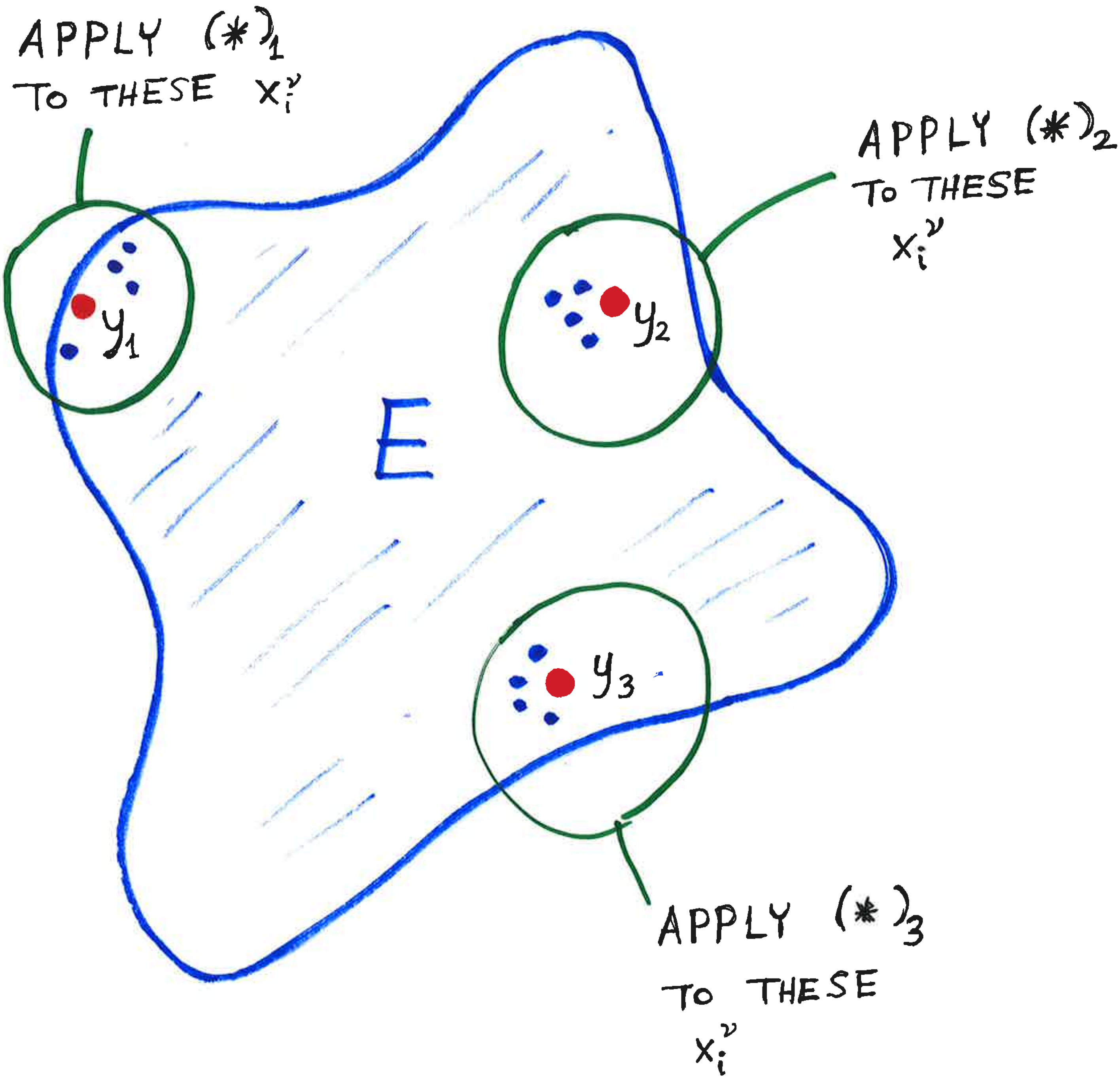
Then we can associate polys

$P_1^\nu, \dots, P_k^\nu$ to the points $x_1^\nu, \dots, x_k^\nu$,

by applying $(*)_q$ to associate

polys to those x_i^ν that lie close to y_q

(Each $q = 1, \dots, S$).


APPLY (*)1
TO THESE x_i^ν
APPLY (*)2
TO THESE
x_i^ν
y1
y2
E
y3
APPLY (*)3
TO THESE
x_i^ν

CLAIM:

For some $M < \infty$

independent of ν,

we have

$$\boxed{\begin{array}{l} |\partial^\alpha P_i^\nu(x_i^\nu)| \leq M \\ \qquad \text{for } |\alpha| \leq m, \quad i = 1, \dots, k \\ \text{and} \\ |\partial^\alpha(P_i^\nu - P_j^\nu)(x_i^\nu)| \leq M |x_i^\nu - x_j^\nu|^{m-|\alpha|} \\ \qquad \text{for } |\alpha| \leq m, \quad i,j = 1, \dots, k. \end{array}}$$

WHY ?

$$\boxed{|\partial^\alpha P_i^\nu(x_i^\nu)| \le M}$$

INDEP. OF ν

because (for some q)

$$|\partial^\alpha (P_i^\nu - \mathring{P}_q)(y_q)| \le |x_i^\nu - y_q|^{m-|\alpha|}$$

by $(*)_q$

$$\boxed{\left|\partial^\alpha (P_i^\nu - P_j^\nu)(x_i^\nu)\right| \leq \left|x_i^\nu - x_j^\nu\right|^{m-|\alpha|}}$$

if x_i^ν & x_j^ν lie close

to the same y_q,

thanks to $(*)_q$.

$$\boxed{|\partial^\alpha(P_i^\nu - P_j^\nu)(x_i^\nu)| \le M\,|x_i^\nu - x_j^\nu|^{m-|\alpha|}}$$

INDEP. OF ν

if x_i^ν & x_j^ν lie close to

y_q, $y_{q'}$ respectively, with $q \ne q'$,

SIMPLY BECAUSE IN THAT CASE

$$|x_i^\nu - x_j^\nu| > \tfrac{1}{2}|y_q - y_{q'}|$$

& WE KNOW THAT

$$|\partial^\alpha P_i^\nu(x_i^\nu)|,\ |\partial^\alpha P_j^\nu(x_j^\nu)| \le M.$$

SO NOW WE KNOW THAT

FOR ν LARGE ENOUGH,

THERE EXIST

$$P_1^\nu \in H(x_1^\nu), \ldots, P_k^\nu \in H(x_k^\nu)$$

S. T.

$$|\partial^\alpha P_i^\nu(x_i^\nu)| \leq M$$

and

$$|\partial^\alpha (P_i^\nu - P_j^\nu)(x_i^\nu)| \leq M |x_i^\nu - x_j^\nu|^{m-|\alpha|},$$

WITH $M < \infty$ INDEP. OF ν.

On the other hand, we picked

$S^\nu = \{x_1^\nu, \cdots, x_k^\nu\}$ and $M_\nu \to \infty$ s.t.

THERE <u>DO NOT</u> EXIST

$P_1 \in H(x_1^\nu), \cdots, P_k \in H(x_k^\nu)$ s.t.

$$|\partial^\alpha P_i(x_i^\nu)| \le M_\nu$$

and

$$|\partial^\alpha(P_i - P_j)(x_i^\nu)| \le M_\nu |x_i^\nu - x_j^\nu|^{m-|\alpha|}.$$

Picking ν large enough

that $M_\nu > M$,

we obtain a CONTRADICTION,

completing the proof that

$\|\mathcal{H}\| < \infty$.

SUMMARY

Whitney's Classic Problem

(Does $\varphi : E \to \mathbb{R}$ EXTEND ?)

is a special case of the

GENERALIZED WHITNEY PROBLEM

for BUNDLES

(Does $\mathcal{H}$ have a section ?)

- GIVEN A BUNDLE $\mathcal{H}$, WE MAY COMPUTE ITS STABLE GLAESER REFINEMENT $\mathcal{H}_*$.

- The bundles $\mathcal{H}$ and $\mathcal{H}_*$ have the same sections.

- $\mathcal{H}_*$ is GLAESER STABLE.

A GLAESER STABLE BUNDLE

HAS A SECTION IF & ONLY IF

ALL ITS FIBERS ARE NON-EMPTY.

The QUANTITATIVE

MAIN THM.

tells us how small we can take

the C^m norm of a SECTION,

if a SECTION EXISTS.

P.S. So far, we've dealt with SCALAR-VALUED functions.

Today's results all generalize to VECTOR-VALUED functions.

That's not just generalization for its own sake, as we'll see in a later lecture.

NEXT TIME, WE SKETCH THE PROOF OF THE

QUANTITATIVE MAIN THM.

APPENDIX :

PROOF OF THE GLAESER STABILIZATION LEMMA

RECALL THE
STATEMENT OF THE LEMMA:

Let $\mathcal{H}_0 = (H_0(x))_{x \in E}$ be a bundle, and let $\mathcal{H}_\ell = (H_\ell(x))_{x \in E}$ be its $\ell^{\underline{th}}$ iterated Glaeser ref.

Then

$$\mathcal{H}_\ell = \mathcal{H}_{2\dim \mathcal{P}+1}$$

for all $\ell \geq 2\dim \mathcal{P} + 1$.

To prove the LEMMA, we will use only two properties of the Glaeser refinement.

(1) If two bundles $\mathcal{H} = (H(x))_{x \in E}$ and $\hat{\mathcal{H}} = (\hat{H}(x))_{x \in E}$ agree in a nbd. of a point $x_0 \in E$, then their Glaeser refinements also agree in a nbd. of x_0.

Let $\mathcal{H} = (H(x))_{x \in E}$ be a bundle, and let $\tilde{\mathcal{H}} = (\tilde{H}(x))_{x \in E}$ be its Glaeser refinement.

2 Then

$$\dim \tilde{H}(x) \leq \liminf_{\substack{y \to x \\ (y \in E)}} \dim H(y).$$

PROPERTIES 1 AND 2 FOLLOW EASILY FROM THE DEF. OF GLAESER REF.

USING 1 and 2, WE WILL PROVE THE FOLLOWING, BY INDUCTION on $l \geq 0$.

$$\boxed{\begin{array}{l} \text{Let } x_0 \in E. \text{ Suppose that} \\ \dim H_{2l+1}(x_0) \geq \dim \mathcal{P} - l. \\ \text{Then } H_q(x_0) = H_{2l+1}(x_0) \\ \text{for all } q \geq 2l+1. \end{array}}$$

$(I)_l$.

The Glaeser Stabilization Lemma follows at once, by taking $l = \dim \mathcal{P}$.

CARRYING OUT THE INDUCTION ON ℓ

BASE CASE $\ell = 0$:

In the BASE CASE $\ell = 0$, (I) asserts that

$$\boxed{\begin{gathered} \dim H_1(x_0) \geq \dim \mathcal{P} \\ \Downarrow \\ H_q(x_0) = H_1(x_0) \text{ for all } q \geq 1. \end{gathered}}$$

If $\dim H_1(x_0) \geq \dim \mathcal{P}$,

then $\liminf\limits_{\substack{y \to x_0 \\ y \in E}} \dim H_0(y) \geq \dim \mathcal{P}$,

yet each $H_0(y)$ is an affine subspace of $\mathcal{P}$.

Consequently, $H_0(y) = \wp$

for all $y \in E$ in a nbd. of x_0.

It follows easily that $H_q(y) = \wp$

for all $y \in E$ in a nbd. of x_0 (all q).

That proves (I) in the

BASE CASE $\ell = 0$.

INDUCTION STEP:

Fix $\ell \geq 0$, and assume that (I) holds for ℓ. We will prove (I) for $\ell+1$. That is, we prove:

$$\boxed{\text{Suppose } \dim H_{2\ell+3}(x_0) \geq \dim \mathcal{P} - \ell - 1. \text{ Then } H_q(x_0) = H_{2\ell+3}(x_0) \text{ for all } q \geq 2\ell+3.}$$

To establish this, we prove the following stronger assertion.

$$\boxed{\begin{array}{l}\text{If } \dim H_{2\ell+3}(x_0) \geq \dim \mathcal{P} - \ell - 1, \\ \text{then} \\ H_{2\ell+1}(y) = H_{2\ell+2}(y) \\ \text{for all } y \in E \text{ close enough to } x_0.\end{array}}$$

$$\uparrow$$

$$(\mathrm{II})$$

So if we can prove (II), then our induction is complete, & the GLAESER STABILIZATION LEMMA follows.

Suppose (II) fails. Then for a sequence $(y_\nu)_{\nu=1,2,\dots}$ of pts of E converging to x_0, we have

$$\dim H_{2\ell+2}(y_\nu) < \dim H_{2\ell+1}(y_\nu).$$

By our INDUCTION HYP. ((I) holds for ℓ) we have

$$\dim H_{2\ell+1}(y_\nu) < \dim \mathcal{P} - \ell.$$

The last two inequalities imply that

$$\dim H_{2\ell+2}(y_\nu) \leq \dim \mathcal{P} - \ell - 2.$$

BUT THEN WE HAVE

$$\dim H_{2\ell+3}(x_0) \leq \liminf_{\nu\to\infty} \dim H_{2\ell+2}(y_\nu)$$

$$\leq \dim \mathcal{P} - \ell - 2,$$

CONTRADICTING OUR ASSUMPTION that

$$\dim H_{2\ell+3}(x_0) \geq \dim \mathcal{P} - \ell - 1.$$

This contradiction proves (II), and with it the GLAESER STABILIZATION LEMMA.

THANK

YOU !